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How to find the conserved quantities of nonlinear discrete equations

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Abstract

A method of finding conserved quantities of nonlinear ordinary difference equations is briefly discussed. The method is based on the hypothesis that the conserved quantities are expressed by a ratio of two polynomials of dependent variables. We obtain the conserved quantities of the 'discrete equation of motion of an anharmonic oscillator' and 'a nonlinear difference equation of third order'.

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1. Introduction

Discretization of integrable systems has been the focus of intense research activity. Typical soliton equations have been discretized in a series of papers [1–5]. Discretization of integrable nonlinear ordinary differential equations has been studied in [6–12]. Recent progress in integrable discrete systems has uncovered remarkable relationships in otherwise unrelated areas of research such as numerical algorithms [13], discrete geometry [14], cellular automaton [15], and quantum integrable systems [16].

However, no systematic method of calculating the conserved quantities of the ordinary nonlinear difference equations has been known. We present a method of calculating the conserved quantities which is based on the hypothesis that the conserved quantities are expressed by a ratio of two polynomials of dependent variables. We explain the method taking the following equations as examples: 'the discrete equation of motion of an anharmonic oscillator' and 'a nonlinear difference equation of third order'.

2. Discretization of an anharmonic oscillator

We consider an equation of motion of an anharmonic oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + ax + bx^3 = 0 \qquad a, b > 0. \tag{1}$$

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We discretize the equation of motion using the bilinear transformation method [6, 11, 12].

Let $x(t) = g(t)/f(t)$. Then equation (1) is transformed into the following form:

$$[D_t^2 g \cdot f + agf]f - g[D_t^2 f \cdot f - bg^2] = 0 \tag{2}$$

where the bilinear operator $D_t^2 g \cdot f$ is defined by

$$D_t^2 g \cdot f = \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} g - 2\frac{\mathrm{d}f}{\mathrm{d}t}\frac{\mathrm{d}g}{\mathrm{d}t} + f\frac{\mathrm{d}^2 g}{\mathrm{d}t^2}. \tag{3}$$

Note that equation (2) is invariant under the following gauge transformation:

$$f(t) \to f(t) \exp \int^t h(t)\,\mathrm{d}t \tag{4}$$

$$g(t) \to g(t) \exp \int^t h(t)\,\mathrm{d}t \tag{5}$$

where $h(t)$ is an arbitrary function of t. Equation (2) is decoupled to the bilinear equations by introducing an arbitrary function $\gamma(t)$,

$$D_t^2 g \cdot f + agf = \gamma(t) gf \tag{6}$$

$$D_t^2 f \cdot f - bg^2 = \gamma(t) f^2. \tag{7}$$

We may choose $\gamma(t) = 0$ by the gauge transformation (4) and (5) with $h(t) = \gamma(t)/2$. Hence equation (1) is transformed into the bilinear form

$$D_t^2 g \cdot f + agf = 0 \tag{8}$$

$$D_t^2 f \cdot f - bg^2 = 0. \tag{9}$$

We discretize the bilinear equation taking the gauge invariance and the time reversibility of the equation into account. First we replace the bilinear operators in equations (6) and (7) by the corresponding bilinear difference operators

$$D_t^2 g \cdot f \to \Delta_t^2 g(t) \cdot f(t) \tag{10}$$

$$D_t^2 f \cdot f \to \Delta_t^2 f(t) \cdot f(t) \tag{11}$$

where

$$\Delta_t^2 g(t) \cdot f(t) \equiv [g(t+\delta) f(t-\delta) - 2g(t) f(t) + g(t-\delta) f(t+\delta)]/\delta^2 \tag{12}$$

$$\Delta_t^2 f(t) \cdot f(t) \equiv 2[f(t+\delta) f(t-\delta) - f(t) f(t)]/\delta^2 \tag{13}$$

δ being a time interval. We assume that the discrete bilinear equations are invariant under the *exponential* gauge transformation

$$f(t) \to f(t) \exp ct \tag{14}$$

$$g(t) \to g(t) \exp ct \tag{15}$$

c being a constant of t. We consider the following discrete bilinear equations:

$$\Delta_t^2 g(t) \cdot f(t) + a\{\gamma_{11} g(t) f(t) + \gamma_{12}[g(t+\delta) f(t-\delta) + g(t-\delta) f(t+\delta)]\} = 0 \tag{16}$$

$$\Delta_t^2 f(t) \cdot f(t) - b\{\gamma_{21} f(t+\delta) f(t-\delta) + \gamma_{22} f(t)^2\} = 0 \tag{17}$$

which are invariant under the gauge transformations (14) and (15), where γ_{11}, γ_{12}, γ_{21} and γ_{22} are free parameters satisfying the relations

$$\gamma_{11} + 2\gamma_{12} = 1 \qquad \gamma_{21} + \gamma_{22} = \gamma.$$

The discrete bilinear forms are transformed, through the dependent variable transformation $g(t) = x(t) f(t)$, into

$$[x(t+\delta) + x(t-\delta)](1 + \delta^2 a\gamma_{12}) f(t+\delta) f(t-\delta) = (2 - \delta^2 a\gamma_{11}) x(t) f^2(t) \tag{18}$$

$$[2 - \delta^2 b\gamma_{21} x(t+\delta) x(t-\delta)] f(t+\delta) f(t-\delta) = [2 + \delta^2 b\gamma_{22} x^2(t)] f^2(t) \tag{19}$$

which are combined to give

$$\frac{[x(t+\delta)+x(t-\delta)](1+\delta^2 a\gamma_{12})}{2-\delta^2 b\gamma_{21}x(t+\delta)x(t-\delta)} = \frac{[2-\delta^2 a\gamma_{11}]x(t)}{2+\delta^2 b\gamma_{22}x^2(t)}. \tag{20}$$

Let $t = n\delta$, n being integers. Then equation (20) is written as

$$\begin{aligned} x_{n+1} - 2x_n + x_{n-1} + a[c_{11}x_n + c_{12}(x_{n+1} + x_{n-1})] \\ +b[c_{21}x_{n+1}x_n x_{n-1} + c_{22}x_n^2(x_{n+1} + x_{n-1})] = 0 \end{aligned} \tag{21}$$

where

$$c_{11} = \delta^2\gamma_{11} \tag{22}$$

$$c_{12} = \delta^2\gamma_{12} \tag{23}$$

$$c_{21} = \delta^2(1 - \tfrac{1}{2}\delta^2 a\gamma_{11})\gamma_{21} \tag{24}$$

$$c_{22} = \delta^2\tfrac{1}{2}(1 + \delta^2 a\gamma_{12})\gamma_{22}. \tag{25}$$

Equation (21) gives an explicit equation of motion

$$x_{n+1} = \frac{\hat{f}_1(x_n) - x_{n-1}\hat{f}_2(x_n)}{\hat{f}_2(x_n) - x_{n-1}\hat{f}_3(x_n)} \tag{26}$$

where

$$\hat{f}_1(x_n) = c_{11}x_n \tag{27}$$

$$\hat{f}_2(x_n) = c_{12} + bc_{22}x_n^2 \tag{28}$$

$$\hat{f}_3(x_n) = -bc_{21}x_n \tag{29}$$

which has the same form as that of the QRT system [17],

$$x_{n+1} = \frac{f_1(x_n) - x_{n-1}f_2(x_n)}{f_2(x_n) - x_{n-1}f_3(x_n)}. \tag{30}$$

The QRT system is known to exhibit a conserved quantity,

$$H = \frac{N}{D} \qquad (x = x_{n-1}, y = x_n) \tag{31}$$

$$N = a_{11}x^2y^2 + a_{12}x^2y + a_{13}x^2 + a_{21}xy^2 + a_{22}xy + a_{23}x + a_{31}y^2 + a_{32}y + a_{33} \tag{32}$$

$$D = b_{11}x^2y^2 + b_{12}x^2y + b_{13}x^2 + b_{21}xy^2 + b_{22}xy + b_{23}x + b_{31}y^2 + b_{32}y + b_{33} \tag{33}$$

provided that $f_1(x)$, $f_2(x)$ and $f_3(x)$ are expressed by

$$\begin{aligned} f_1(x) = (a_{21}x^2 + a_{22}x + a_{23})(b_{31}x^2 + b_{32}x + b_{33}) \\ -(a_{31}x^2 + a_{32}x + a_{33})(b_{21}x^2 + b_{22}x + b_{23}) \end{aligned} \tag{34}$$

$$\begin{aligned} f_2(x) = (a_{31}x^2 + a_{32}x + a_{33})(b_{11}x^2 + b_{12}x + b_{13}) \\ -(a_{11}x^2 + a_{12}x + a_{13})(b_{31}x^2 + b_{32}x + b_{33}) \end{aligned} \tag{35}$$

$$\begin{aligned} f_3(x) = (a_{11}x^2 + a_{12}x + a_{13})(b_{21}x^2 + b_{22}x + b_{23}) \\ -(a_{21}x^2 + a_{22}x + a_{23})(b_{11}x^2 + b_{12}x + b_{13}). \end{aligned} \tag{36}$$

However, it is not easy to find relations between our $\hat{f}_1$, $\hat{f}_2$, $\hat{f}_3$ and their f_1, f_2, f_3. We shall return to this point after finding the conserved quantity of equation (26).

Finally we note that the discrete equation (26) is reduced, in the limit of small δ, to the equation of motion of the anharmonic oscillator

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + ax + bx^3 = 0. \tag{37}$$

3. Conserved quantity of the discrete equation of motion of the anharmonic oscillator

We conjecture that the conserved quantity H_n of the discrete equation of motion of the anharmonic oscillator is expressed with a ratio of polynomials of variables x_n and x_{n-1}, which has the following properties:

(i) The discrete equation is invariant under the transformation

$$x_{n+j} \to x_{n-j} \qquad j = \pm 1$$

which implies the time reversibility of the equation. Accordingly the conserved quantity is assumed to be a symmetric function of x_n and x_{n-1}.

(ii) The discrete equation is explicit. Hence cubic terms such as x_n^3 and x_{n-1}^3 are excluded from the H_n.

Hence we have the conserved quantity H_n expressed with

$$H_n = \frac{a_0 + a_1 p_1(n) + \cdots + a_5 p_5(n)}{b_0 + b_1 p_1(n) + \cdots + b_5 p_5(n)} \tag{38}$$

where

$$p_1(n) = x_n + x_{n-1} \tag{39}$$

$$p_2(n) = x_n x_{n-1} \tag{40}$$

$$p_3(n) = x_n^2 + x_{n-1}^2 \tag{41}$$

$$p_4(n) = x_n^2 x_{n-1} + x_n x_{n-1}^2 \tag{42}$$

$$p_5(n) = x_n^2 x_{n-1}^2. \tag{43}$$

We rewrite the above form as

$$1 + c_1 p_1(n) + c_2 p_2(n) + \cdots + c_5 p_5(n) = 0 \tag{44}$$

where

$$c_j = (a_j - H_n b_j)/(a_0 - H_n b_0) \qquad \text{for} \quad j = 1, 2, \ldots, 5. \tag{45}$$

We note that all c_j are conserved quantities, which depend only on the initial values of the discrete equation. Accordingly we have a set of linear equations for $c_1, c_2, \ldots, c_5$

$$1 + c_1 p_1(n) + c_2 p_2(n) + \cdots + c_5 p_5(n) = 0 \tag{46}$$

$$1 + c_1 p_1(n+1) + c_2 p_2(n+1) + \cdots + c_5 p_5(n+1) = 0 \cdots\cdots \tag{47}$$

$$1 + c_1 p_1(n+4) + c_2 p_2(n+4) + \cdots + c_5 p_5(n+4) = 0. \tag{48}$$

Solving the linear equations we obtain

$$c_j(n) = \begin{vmatrix} p_1(n) & p_2(n) & \cdots & -1 & \cdots & p_5(n) \\ p_1(n+1) & p_2(n+1) & \cdots & -1 & \cdots & p_5(n+1) \\ p_1(n+2) & p_2(n+2) & \cdots & -1 & \cdots & p_5(n+2) \\ p_1(n+3) & p_2(n+3) & \cdots & -1 & \cdots & p_5(n+3) \\ p_1(n+4) & p_2(n+4) & \cdots & -1 & \cdots & p_5(n+4) \end{vmatrix} \Bigg/ \Delta(n) \tag{49}$$

for $j = 1, 2, \ldots, 5$, where

$$\Delta(n) = \begin{vmatrix} p_1(n) & p_2(n) & \cdots & p_5(n) \\ p_1(n+1) & p_2(n+1) & \cdots & p_5(n+1) \\ p_1(n+2) & p_2(n+2) & \cdots & p_5(n+2) \\ p_1(n+3) & p_2(n+3) & \cdots & p_5(n+3) \\ p_1(n+4) & p_2(n+4) & \cdots & p_5(n+4) \end{vmatrix}. \tag{50}$$

The assumption that all $c_j(n)$ are conserved quantities, namely

$$c_j(n+1) - c_j(n) = 0 \qquad \text{for} \quad j = 1, 2, \ldots, 5 \tag{51}$$

is transformed into the following condition:

$$\begin{vmatrix} 1 & p_1(n) & p_2(n) & \cdots & p_5(n) \\ 1 & p_1(n+1) & p_2(n+1) & \cdots & p_5(n+1) \\ 1 & p_1(n+2) & p_2(n+2) & \cdots & p_5(n+2) \\ 1 & p_1(n+3) & p_2(n+3) & \cdots & p_5(n+3) \\ 1 & p_1(n+4) & p_2(n+4) & \cdots & p_5(n+4) \\ 1 & p_1(n+5) & p_2(n+5) & \cdots & p_5(n+5) \end{vmatrix} = 0. \tag{52}$$

Because the numerator of $c_j(n+1) - c_j(n)$ for $j = 1$, for example, is

$$\begin{vmatrix} -1 & p_2(n+1) & \cdots & p_5(n+1) \\ -1 & p_2(n+2) & \cdots & p_5(n+2) \\ -1 & p_2(n+3) & \cdots & p_5(n+3) \\ -1 & p_2(n+4) & \cdots & p_5(n+4) \\ -1 & p_2(n+5) & \cdots & p_5(n+5) \end{vmatrix} \begin{vmatrix} p_1(n) & p_2(n) & \cdots & p_5(n) \\ p_1(n+1) & p_2(n+1) & \cdots & p_5(n+1) \\ p_1(n+2) & p_2(n+2) & \cdots & p_5(n+2) \\ p_1(n+3) & p_2(n+3) & \cdots & p_5(n+3) \\ p_1(n+4) & p_2(n+4) & \cdots & p_5(n+4) \end{vmatrix}$$

$$- \begin{vmatrix} -1 & p_2(n) & \cdots & p_5(n) \\ -1 & p_2(n+1) & \cdots & p_5(n+1) \\ -1 & p_2(n+2) & \cdots & p_5(n+2) \\ -1 & p_2(n+3) & \cdots & p_5(n+3) \\ -1 & p_2(n+4) & \cdots & p_5(n+4) \end{vmatrix}$$

$$\times \begin{vmatrix} p_1(n+1) & p_2(n+1) & \cdots & p_5(n+1) \\ p_1(n+2) & p_2(n+2) & \cdots & p_5(n+2) \\ p_1(n+3) & p_2(n+3) & \cdots & p_5(n+3) \\ p_1(n+4) & p_2(n+4) & \cdots & p_5(n+4) \\ p_1(n+5) & p_2(n+5) & \cdots & p_5(n+5) \end{vmatrix}$$

which is reduced, by the Jacobi identity of determinants, to

$$= \begin{vmatrix} 1 & p_1(n) & p_2(n) & \cdots & p_5(n) \\ 1 & p_1(n+1) & p_2(n+1) & \cdots & p_5(n+1) \\ 1 & p_1(n+2) & p_2(n+2) & \cdots & p_5(n+2) \\ 1 & p_1(n+3) & p_2(n+3) & \cdots & p_5(n+3) \\ 1 & p_1(n+4) & p_2(n+4) & \cdots & p_5(n+4) \\ 1 & p_1(n+5) & p_2(n+5) & \cdots & p_5(n+5) \end{vmatrix} \begin{vmatrix} p_2(n+1) & \cdots & p_5(n+1) \\ p_2(n+2) & \cdots & p_5(n+2) \\ p_2(n+3) & \cdots & p_5(n+3) \\ p_2(n+4) & \cdots & p_5(n+4) \end{vmatrix}.$$

Accordingly $c_j(n+1) = c_j(n)$, and $c_j(n)$ are conserved quantities if equation (52) holds.

In general we conjecture, for a given nonlinear discrete equation of order $k+1$

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}) \tag{53}$$

that a conserved quantity H_n is given by

$$H_n = \frac{a_0 + a_1 p_1(n) + \cdots + a_N p_N(n)}{b_0 + b_1 p_1(n) + \cdots + b_N p_N(n)} \tag{54}$$

where

$$p_j(n) = p_j(x_n, x_{n-1}, \ldots, x_{n-k}) \qquad \text{for} \quad j = 1, 2, \ldots, N. \tag{55}$$

Then we have linear equations for $c_j = (a_j - H_n)/(a_0 - H_n b_0)$,

$$c_1 p_1(n+l) + c_2 p_2(n+l) + \cdots + c_N p_N(n+l) = -1 \qquad \text{for all } l. \tag{56}$$

The following quantities:

$$c_j(n) = \frac{\begin{vmatrix} p_1(n) & p_2(n) & \cdots & -1 & \cdots & p_N(n) \\ p_1(n+1) & p_2(n+1) & \cdots & -1 & \cdots & p_N(n+1) \\ p_1(n+2) & p_2(n+2) & \cdots & -1 & \cdots & p_N(n+2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_1(n+N-1) & p_2(n+N-1) & \cdots & -1 & \cdots & p_N(n+N-1) \end{vmatrix}}{\begin{vmatrix} p_1(n) & p_2(n) & \cdots & p_j(n) & \cdots & p_N(n) \\ p_1(n+1) & p_2(n+1) & \cdots & p_j(n+1) & \cdots & p_N(n+1) \\ p_1(n+2) & p_2(n+2) & \cdots & p_j(n+2) & \cdots & p_N(n+2) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_1(n+N-1) & p_2(n+N-1) & \cdots & p_j(n+N-1) & \cdots & p_N(n+N-1) \end{vmatrix}}$$

for $j = 1, 2, \ldots, N$ are the conserved quantities of the discrete equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}) \qquad \text{for integers } n$$

provided that

$$\begin{vmatrix} 1 & p_1(n) & p_2(n) & \cdots & p_N(n) \\ 1 & p_1(n+1) & p_2(n+1) & \cdots & p_N(n+1) \\ 1 & p_1(n+2) & p_2(n+2) & \cdots & p_N(n+2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & p_1(n+N) & p_2(n+N) & \cdots & p_N(n+N) \end{vmatrix} = 0. \tag{57}$$

All conserved quantities of the discrete equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k})$$

are, in principle, obtained by checking the condition (57). However, in practice, it is not easy to check the condition for $N > 5$ by using a computer at hand. In order to minimize a number of unknown parameters, we evaluate them numerically at first. Returning to equation (44), we solve it numerically and find that $c_1 = c_4 = 0$. Thus the linear equations to be solved are reduced to

$$c_2 p_1(n+l) + c_3 p_2(n+l) + c_5 p_5(n+l) = -1 \qquad \text{for all } l \tag{58}$$

where

$$p_2(n) = x_n x_{n-1} \tag{59}$$

$$p_3(n) = x_n^2 + x_{n-1}^2 \tag{60}$$

$$p_5(n) = x_n^2 x_{n-1}^2. \tag{61}$$

Now it is easy to check the condition

$$\begin{vmatrix} 1 & p_2(n) & p_3(n) & p_5(n) \\ 1 & p_2(n+1) & p_3(n+1) & p_5(n+1) \\ 1 & p_2(n+2) & p_3(n+2) & p_5(n+2) \\ 1 & p_2(n+3) & p_3(n+3) & p_5(n+3) \end{vmatrix} = 0. \tag{62}$$

Solving equation (58) for $l = -1, 0, 1$ we find that one of the conserved quantities is expressed by

$$H_0 = \frac{N_0}{D_0} \tag{63}$$

$$N_0 = c_{11}[c_{12}(x_n^2 + x_{n-1}^2) - c_{11} x_n x_{n-1}] + b(c_{11}c_{22} + c_{12}c_{21}) x_n^2 x_{n-1}^2 \tag{64}$$

$$D_0 = c_{11}[c_{11} + bc_{21}(x_n^2 + x_{n-1}^2)] + b^2 c_{21}^2 x_n^2 x_{n-1}^2 \tag{65}$$

which is the conserved quantity of the discrete equation of motion of the anharmonic oscillator equation (26). The rank of the Jacobi matrix shows that other conserved quantities c_2, c_3 and c_5 are functionally dependent on H_0. Here we remark the relation to the QRT system. Comparing our conserved quantity H_0 with H of the QRT system we find

$$a_{11} = b(c_{11}c_{22} + c_{12}c_{21}) \qquad a_{12} = 0 \qquad a_{13} = c_{11}c_{12} \tag{66}$$
$$a_{21} = 0 \qquad a_{22} = -c_{11}^2 \qquad a_{23} = 0 \tag{67}$$
$$a_{31} = a_{13} \qquad a_{32} = 0 \qquad a_{33} = 0 \tag{68}$$
$$b_{11} = b^2c_{21}^2 \qquad b_{12} = 0 \qquad b_{13} = bc_{11}c_{21} \tag{69}$$
$$b_{21} = 0 \qquad b_{22} = 0 \qquad b_{23} = 0 \tag{70}$$
$$b_{31} = bc_{11}c_{21} \qquad b_{32} = 0 \qquad b_{23} = c_{11}^2. \tag{71}$$

Accordingly we find that $f_1(x)$, $f_2(x)$ and $f_3(x)$ of the QRT system are expressed with $\hat{f}_1(x_n)$, $\hat{f}_2(x_n)$ and $\hat{f}_3(x_n)$ of the discrete equation of motion as follows:

$$f_1(x) = -c_{11}^2(c_{11} + bc_{21}x_n^2)\hat{f}_1(x_n) \tag{72}$$
$$f_2(x) = -c_{11}^2(c_{11} + bc_{21}x_n^2)\hat{f}_2(x_n) \tag{73}$$
$$f_3(x) = -c_{11}^2(c_{11} + bc_{21}x_n^2)\hat{f}_3(x_n) \tag{74}$$

which shows that the discrete equation of motion (21) is one of the QRT system. We note that Suris [7] has studied a discrete equation which is expressed as a QRT system of the special form

$$x_{n+1} = \frac{g_1(x_n) - x_{n-1}g_2(x_n)}{g_2(x_n) - x_{n-1}g_3(x_n)} \tag{75}$$

where

$$g_1(x_n) = a + (2 - 2e + b)x_n + cx_n^2/3 \tag{76}$$
$$g_2(x_n) = 1 - (e + cx_n/3 + dx_n^2/2) \tag{77}$$
$$g_3(x_n) = 0 \tag{78}$$

where a, b, c, d, e are constant parameters. The conserved quantity is a polynomial of the dependent variables:

$$\hat{H}(n) = \tfrac{1}{2}(x_n - x_{n-1})^2 - \tfrac{1}{2}a(x_n + x_{n-1})^2 \\ -\tfrac{1}{2}bx_nx_{n-1} - \tfrac{1}{6}cx_nx_{n-1}(x_n + x_{n-1}) - \tfrac{1}{4}x_n^2x_{n-1}^2 - \tfrac{1}{2}e(x_n - x_{n-1})^2. \tag{79}$$

4. Conserved quantities of the difference equation of third order

We take the following third-order difference equation as the next example:

$$x_{n+2} = (1 + x_n + x_{n+1})/x_{n-1} \tag{80}$$

which is known to show ‘a recurrence of period 8’ [18], namely the system returns to the initial state after mapping eight times.

The discrete equation (80) is of the third order and is time reversible. Accordingly the conserved quantity H_n is a function of x_{n+1}, x_n, x_{n-1} and is symmetric with respect to x_{n+1} and x_{n-1}. We assume the conserved quantity of the following form:

$$H_n = \frac{a_0 + a_1p_1(n) + \cdots + a_{13}p_{13}(n)}{b_0 + b_1p_1(n) + \cdots + b_{13}p_{13}(n)} \tag{81}$$

with

$$p_1(n) = x(n) \tag{82}$$

$$p_2(n) = x(n+1) + x(n-1) \tag{83}$$

$$p_3(n) = x(n)^2 \tag{84}$$

$$p_4(n) = x(n)(x(n+1) + x(n-1)) \tag{85}$$

$$p_5(n) = x(n+1)^2 + x(n-1)^2 \tag{86}$$

$$p_6(n) = x(n+1)x(n-1) \tag{87}$$

$$p_7(n) = x(n)^2(x(n+1) + x(n-1)) \tag{88}$$

$$p_8(n) = x(n)(x(n+1)^2 + x(n-1)^2) \tag{89}$$

$$p_9(n) = x(n+1)x(n-1)(x(n+1) + x(n-1)) \tag{90}$$

$$p_{10}(n) = x(n)^2(x(n+1)^2 + x(n-1)^2) \tag{91}$$

$$p_{11}(n) = x(n)^2 x(n+1)x(n-1) \tag{92}$$

$$p_{12}(n) = x(n)x(n+1)x(n-1)(x(n+1) + x(n-1)) \tag{93}$$

$$p_{13}(n) = x(n+1)^2 x(n-1)^2 \tag{94}$$

$x(n)$ being x_n for any n. The number N of unknown parameters c_j of the equation is greater than ten and it is not easy to check the condition (57).

We follow a different approach to the problem. We note that equation (80) is singular at $x_n = 0$:

$$x_{n+2} \to \infty \qquad \text{as} \quad x_{n-1} \to 0.$$

Taking this property of the equation into account, we assume that the denominator of the conserved quantity $H(n)$ is of the form $(x_{n+1}x_nx_{n-1})^n$, n being an integer. Hence a trial form of H_n is

$$H_n = \left[\sum_{j=0}^{13} c(j)p_j(n)\right] \Big/ [x_{n+1}x_nx_{n-1}]. \tag{95}$$

The unknown parameters $c(j)$, for $j = 0, 1, 2, \ldots, 13$ are determined by the equation

$$H_{n+1} - H_n = 0 \tag{96}$$

which gives

$$\begin{array}{lllll} c(1) = 2 & c(2) = 2 & c(3) = 1 & c(4) = 3 & c(5) = 1 \\ c(6) = 3 & c(7) = 1 & c(8) = 1 & c(9) = 1 & c(10) = 0 \\ c(11) = 1 & c(12) = 1 & c(13) = 0 & & \end{array}$$

where we have chosen an arbitrary parameter $c(0) = 1$. Accordingly one of the conserved quantities is given by

$$\begin{aligned} H_n^{(1)} = [&1 + 2x(n) + 2(x(n+1) + x(n-1)) + x(n)^2 + 3x(n)(x(n+1) + x(n-1)) \\ &+x(n+1)^2 + x(n-1)^2 + 3x(n+1)x(n-1) + \\ &x(n)^2(x(n+1) + x(n-1)) + x(n)(x(n+1)^2 + x(n-1)^2) \\ &+x(n+1)x(n-1)(x(n+1) + x(n-1)) + x(n)^2x(n+1)x(n-1) \\ &+x(n)x(n+1)x(n-1)(x(n+1) + x(n-1))]/[x(n+1)x(n)x(n-1)]. \end{aligned} \tag{97}$$

Another conserved quantity is obtained by including new terms in the trial form of $H_n^{(1)}$:

$$H_n^{(2)} = \left[\sum_{j=0}^{15} c(j)p_j(n)\right] \Big/ [x_{n+1}x_nx_{n-1}] \tag{98}$$

where

$$p_{14}(n) = x(n)^3 \tag{99}$$

$$p_{15}(n) = x(n+1)^3 + x(n-1)^3. \tag{100}$$

The cubic terms such as $x(n+1)^3$, $x(n)^3$ and $x(n-1)^3$ in the numerator of $H_n^{(2)}$ are permissible in this case because they are divided by the denominator $x(n+1)x(n)x(n-1)$. Substituting $H_n^{(2)}$ into equation (96) we determine new parameters $c(j)$, for $j = 0, 1, 2, \ldots, 15$ as follows:

$$\begin{array}{lllll} c(1)=1 & c(2)=0 & c(3)=2 & c(4)=1 & c(5)=0 \\ c(6)=1 & c(7)=1 & c(8)=0 & c(9)=1 & c(10)=0 \\ c(11)=0 & c(12)=0 & c(13)=1 & c(14)=1 & c(15)=0 \end{array}$$

where we have fixed two redundant parameters $c(14)$ and $c(0)$ to be 1 and 0, respectively. Accordingly the second conserved quantity $H_n^{(2)}$ is given by

$$\begin{aligned} H_n^{(2)} = {} & [x(n) + 2x(n)^2 + x(n)(x(n+1) + x(n-1)) + x(n+1)x(n-1) \\ & + x(n)^2(x(n+1) + x(n-1)) + x(n+1)x(n-1)(x(n+1) + x(n-1)) \\ & + x(n+1)^2 x(n-1)^2 + x(n)^3]/[x(n+1)x(n)x(n-1)]. \end{aligned} \tag{101}$$

We have found two conserved quantities, $H_n^{(1)}$ and $H_n^{(2)}$, of the third-order difference equation

$$x_{n+2}x_{n-1} = 1 + x_n + x_{n+1}. \tag{102}$$

The present procedure of finding conserved quantities is applied to other difference equations. In the appendix we give a list of third-order difference equations of the form

$$x_{n+2}x_{n-1} = f(x_n, x_{n+1}) \tag{103}$$

whose conserved quantities are obtained by following the present procedure.

Appendix. Third-order difference equations exhibiting two conserved quantities

We have investigated difference equations of third order of the following special form:

$$x_{n+2}x_{n-1} = f(x_n, x_{n+1}). \tag{104}$$

It is found by numerical simulation that the algebraic entropy [19, 20] of the map shows the polynomial growth of the degree if $f(x_n, x_{n+1})$ is one of the following forms:

$$(Y1) \quad f(x_n, x_{n+1}) = \frac{a_0 + a_1 x_n + a_1 x_{n+1} + a_3 x_n x_{n+1}}{a_3 + b_1 x_n + b_1 x_{n+1} + b_3 x_n x_{n+1}}$$

$$(Y2) \quad f(x_n, x_{n+1}) = \frac{a_0 + a_0 x_n + a_0 x_{n+1} + a_3 x_n x_{n+1}}{a_0 + a_3 x_n + a_3 x_{n+1} + a_3 x_n x_{n+1}}$$

$$(Y3) \quad f(x_n, x_{n+1}) = \frac{-a_0 + a_0 x_n - a_0 x_{n+1} + a_3 x_n x_{n+1}}{a_0 + a_3 x_n - a_3 x_{n+1} - a_3 x_n x_{n+1}}$$

$$(Y4) \quad f(x_n, x_{n+1}) = \frac{a_0 + a_1 x_n + a_1 x_{n+1} + a_1 x_n x_{n+1}}{b_0 + b_0 x_n + b_0 x_{n+1} + b_0 x_n x_{n+1}}$$

$$(Y5) \quad f(x_n, x_{n+1}) = \frac{a_1 x_n - a_1 x_{n+1} + a_3 x_n x_{n+1}}{a_3 - b_1 x_n + b_1 x_{n+1}}$$

$$(Y6) \quad f(x_n, x_{n+1}) = \frac{a_3 x_n x_{n+1}}{b_1 x_n + b_1 x_{n+1} + b_3 x_n x_{n+1}}$$

$$(Y7) \quad f(x_n, x_{n+1}) = \frac{a_0 + a_1 x_n}{a_1 x_n + a_0 x_n x_{n+1}}$$

$$(Y8) \quad f(x_n, x_{n+1}) = \frac{a_0 + a_1 x_n}{-a_1 x_n + a_0 x_n x_{n+1}}$$

$$(Y9) \quad f(x_n, x_{n+1}) = \frac{a_1 x_n + a_1 x_n x_{n+1}}{a_1 + a_1 x_n}$$

where $a_0, a_1, \ldots, b_3$ are constant parameters. Equation (80) is a special case ($a_3 = 0$) of equation (Y2). These equations are transformed, through the transformation $x_n = g_n/f_n$, into the bilinear forms which are invariant under the exponential gauge-transformation,

$$f_n \to f_n \exp(ct) \qquad g_n \to g_n \exp(ct). \tag{105}$$

We have found, following the present procedure, two functionally independent conserved quantities of all equations listed above. The computer output of the conserved quantities can be found at `http://www.hirota.info.waseda.ac.jp/~hirota/conserved.zip`

References

[1] Hirota R 1977 Nonlinear partial difference equations I. A difference analogue of the Korteweg–de Vries equation *J. Phys. Soc. Japan* **43** 1424–33

[2] Hirota R 1977 Nonlinear partial difference equations II. Discrete-time Toda equation *J. Phys. Soc. Japan* **43** 2074–8

[3] Hirota R 1977 Nonlinear partial difference equations III. Discrete sine-Gordon equation *J. Phys. Soc. Japan* **43** 2079–86

[4] Hirota R 1978 Nonlinear partial difference equations IV. Bäcklund transformation for the discrete-time Toda equation *J. Phys. Soc. Japan* **45** 321–32

[5] Hirota R 1981 Discrete analogue of a generalized Toda equation *J. Phys. Soc. Japan* **50** 3785–91

[6] Hirota R 1982 Difference analogues of nonlinear evolution equations in Hamiltonian form *Technical Report* No A-12 Department of Applied Mathematics, Faculty of Engineers, Hiroshima University

[7] Suris Yu B 1989 Integrable mappings of the standard type *Funkt. Anal. Prilozhen* **23** 84–5

[8] Grammaticos B, Ramani A and Papageorgiou V 1991 Do integral mappings have the Painlevé property? *Phys. Rev. Lett.* A **67** 1825–7

[9] Ramani A, Grammaticos B and Hietarinta J 1991 Discrete versions of the Painlevé equations *Phys. Rev. Lett.* A **67** 1829–32

[10] Bobenko A I, Lorbeer B and Suris Yu B 1998 Integral discretization of the Euler top *J. Math. Phys.* **39** 6668–83

[11] Hirota R and Kimura K 2000 Discretization of the Euler top *J. Phys. Soc. Japan* **69** 627–30

[12] Kimura K and Hirota R 2000 Discretization of the Lagrange top *J. Phys. Soc. Japan* **69** 3193–9

[13] Papageorgiou V, Gramaticos B and Ramani A 1993 *Phys. Lett.* A **179** 111

[14] Bobenko A I and Pinkall U 1996 Discrete surfaces with constant negative Gaussian curvature and the Hirota equation *J. Diff. Geom.* **43** 527–611

[15] Tokihiro T, Takahashi D, Matukidaira J and Satsuma J 1997 *J. Phys. Soc. Japan* **43** 2074–8

[16] Zabroding A 1997 Discrete Hirota's equation in quantum integrable models *Int. J. Mod. Phys.* B **11** 3125–58

[17] Quispel G R W, Robert J A G and Thompson C J 1989 Integrable mappings and soliton equations II *Physica* D **34** 183–92

[18] Graham R L, Knuth D E and Patashnik O 1994 *Concrete Mathematics* (Reading, MA: Addison-Wesley) p 501

[19] Hietarinta J and Viallet C 1998 Singularity confinement and chaos in discrete systems *Phys. Rev. Lett.* **81** 325–8

[20] Bellon M P and Viallet C-M 1999 Algebraic entropy *Commun. Math. Phys.* **204** 425–37